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Sylvie Benzoni-Gavage

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Abstract. This paper is concerned with nonlocal generalizations of the inviscid Burgers equation arising as amplitude equations for weakly nonlinear surface waves. Under homogeneity and stability assumptions on the involved kernel it is shown that the Cauchy problem is locally well-posed in $H^2(\mathbb{R})$, and a blow-up criterion is derived. The proof is based on a priori estimates without loss of derivatives, and on a regularization of both the equation and the initial data.

Keywords. Nonlinear surface wave, amplitude equation, smooth solutions, blow-up criterion.

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1 Introduction

We consider a nonlocal generalization of the inviscid Burgers equation

$$\partial_t u + \partial_x \mathcal{Q}[u] = 0\,, \tag{1.1}$$

where $\mathcal{Q}$ is a quadratic nonlocal operator given in Fourier variables by

$$\mathcal{F}\mathcal{Q}[u](k) = \int \Lambda(k-\ell,\ell)\widehat{u}(k-\ell)\widehat{u}(\ell)\mathrm{d}\ell\,, \tag{1.2}$$

when u is Schwartz. (Throughout the paper $\mathcal{F}$ denotes the Fourier transform in one space dimension.) Equations of this type arise in particular as amplitude equations for weakly nonlinear waves [5, 1, 3]. The kernel Λ is piecewise smooth and satisfies the following conditions

(*i*) symmetry: $\Lambda(k,\ell) = \Lambda(\ell,k) \quad \forall k,\ell \in \mathbb{R}\,,$

(*ii*) reality: $\Lambda(-k,-\ell) = \overline{\Lambda(k,\ell)} \quad \forall k,\ell \in \mathbb{R}\,,$

(*iii*) homogeneity: $\Lambda(\alpha k,\alpha\ell) = \Lambda(k,\ell) \quad \forall k,\ell \in \mathbb{R}, \text{and } \alpha > 0\,,$

(*iv*) structure: $\Lambda(k+\xi,-\xi) = \overline{\Lambda(k,\xi)} \quad \forall k,\xi \in \mathbb{R}\,,$

the latter being possibly replaced by

(*v*) stability: $\Lambda(1,0-) = \overline{\Lambda(1,0+)}\,.$

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(When $\Lambda \equiv 1/2$, (1.1) is nothing but the classical inviscid Burgers equation.) The symmetry condition is not actually a restriction: any kernel Λ defining an operator $\mathcal{Q}$ as in (1.2) can be changed into a symmetric one, by change of variables $\ell \mapsto k - \ell$ in the integral. The reality condition ensures that $\mathcal{Q}[u]$ is real valued if u is so (which is equivalent to $\widehat{u}(-k) = \overline{\widehat{u}(k)}$). When (1.1) is an amplitude equation, the homogeneity condition is linked to the scale invariance of surface waves. Homogeneity of degree zero ensures in particular that Λ is bounded, since it is piecewise smooth, and that its singularities occur along rays. We shall assume these singularities are located exclusively on the axes $k = 0$, $\ell = 0$ and on $\{(k, \ell)\,;\ k + \ell = 0\}$, which is clearly compatible with all other assumptions. The one that we call structure condition may look mysterious at first glance. The weaker condition (v) was pointed out by Hunter [5] as formally ensuring the linearized stability of constant states. Alì, Hunter and Parker [1] have observed that (iv) yields a Hamiltonian structure for (1.1). More precisely, if (iv) holds true, (1.1) equivalently reads

$$\partial_t u + \partial_x \delta\mathcal{H}[u] = 0\,, \tag{1.3}$$

where the Hamiltonian is defined by

$$\mathcal{F}\mathcal{H}[u](k) \;=\; \frac{1}{3}\iint \Lambda(k,\xi)\widehat{u}(k)\widehat{u}(\xi)\widehat{u}(-k-\xi)\,\mathrm{d}k\,\mathrm{d}\xi \tag{1.4}$$

if u is Schwartz. In this framework – that is, assuming (iv) – Hunter has shown the local existence of periodic solutions of (1.3) [6]. The purpose of this paper is to show the local existence of smooth solutions to (1.1) on the real line, and more precisely the local well-posedness of (1.1) in $H^2(\mathbb{R})$, assuming the stability condition (v) but not (iv). Indeed, (v) turns out to be sufficient to get *a priori* estimates for (1.1) without loss of derivatives. To prove well-posedness we shall then use a regularization method as proposed by Taylor [7, p.360].

2 A priori estimates

Let us rewrite (1.1) as

$$\partial_t u + 2\mathcal{B}[u, \partial_x u] = 0\,, \tag{2.5}$$

where

$$\mathcal{F}\mathcal{B}[u,w](k) = \int \Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{u}(\ell)\mathrm{d}\ell\,. \tag{2.6}$$

The operator $\mathcal{B}$ is well-defined and bilinear on $\mathscr{S}(\mathbb{R})\times\mathscr{S}(\mathbb{R})$, and it is clearly symmetric because of the symmetry property of Λ. Furthermore, by straightforward inspection,

$$\partial_x\mathcal{B}[u,w] = \mathcal{B}[\partial_x u, w] + \mathcal{B}[u, \partial_x w]\,, \tag{2.7}$$

so that, because also of the symmetry of $\mathcal{B}$, (1.1) and (2.5) are indeed equivalent for smooth solutions.

In addition, the operator $\mathcal{B}$ extends to a continuous operator from $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Indeed, by its definition (2.6), Plancherel's theorem and $L^1 - L^2$ convolution estimates we have

$$\|\mathcal{B}(u,w)\|_{L^2} = \|\mathcal{F}\mathcal{B}(u,w)\|_{L^2} \;\le\; \|\Lambda\|_{L^\infty}\,\|\widehat{u}\|_{L^1}\,\|\widehat{w}\|_{L^2} \;\lesssim\; \|\Lambda\|_{L^\infty}\,\|u\|_{H^1}\,\|w\|_{L^2}\,. \tag{2.8}$$

Here above and throughout the paper, the symbol $\lesssim$ means 'less or equal to a harmless constant times'. In a similar but more symmetric way – distributing derivatives equally on u and w -, we also find that $\mathcal{B}$ is a continuous bilinear operator in $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$, with

$$\|\mathcal{B}(u,w)\|_{H^1} \lesssim \|\Lambda\|_{L^\infty}\,\|u\|_{H^1}\,\|w\|_{H^1}\,, \tag{2.9}$$

$$\|\mathcal{B}(u,w)\|_{H^2} \lesssim \|\Lambda\|_{L^\infty} \left(\|u\|_{H^1} \|w\|_{H^2} + \|u\|_{H^2} \|w\|_{H^1}\right). \tag{2.10}$$

Now we consider, for a given (smooth enough) u, the linear equation

$$\partial_t v + 2\mathcal{B}[u, \partial_x v] = 0\,, \tag{2.11}$$

and we look for *a priori* estimates without loss of derivatives. All functions of the space variable x will be taken with values in $\mathbb{R}$, and we shall use repeatedly the corresponding property $(\widehat{u}(-k) = \overline{\widehat{u}(k)})$ in the Fourier variable.

Lemma 2.1 *We assume that Λ is $\mathscr{C}^1$ outside the lines $k = 0$, $\ell = 0$, and $k + \ell = 0$, and has $\mathscr{C}^1$ continuations to the sectors delimited by these lines. If in addition it satisfies (ii), (iii), (v), then the solutions of* (2.11) *satisfy the following a priori estimates*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2 \le C(\Lambda)\, \|\mathcal{F}(\partial_x u)\|_{L^1}\, \|v\|_{L^2}^2\,, \tag{2.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^1}^2 \le C(\Lambda)\, \|\mathcal{F}(\partial_x u)\|_{L^1}\, \|v\|_{H^1}^2\,, \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^2}^2 \le C(\Lambda)\, \|u\|_{H^2}\, \|v\|_{H^2}^2\,, \tag{2.14}$$

where $C(\Lambda)$ depends only on $\|\Lambda\|_{L^\infty}$, $\|\nabla\Lambda\|_{L^\infty(D)}$, and $\|\nabla\Lambda\|_{L^\infty(I)}$ with

$$D = \{(-1+\theta, -\theta)\,; 0 < \theta < 1\,\}\,, \quad I = \{(1, -\theta)\,; 0 < \theta < 1\,\}\,.$$

Proof. If v is a solution of (2.11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^2}^2 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{v}(t)\|_{L^2}^2 \;=$$

$$-2\,\mathrm{Re} \iint i(k-\ell)\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,.$$

The integral here above can be split into

$$-\iint i\ell\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k$$

$$+\iint ik\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,.$$

By Fubini and the Cauchy-Schwarz inequality the modulus of the first integral is bounded by $\|\Lambda\|_{L^\infty}\, \|\mathcal{F}(\partial_x u)\|_{L^1}\, \|v(t)\|_{L^2}^2$. We now concentrate on the real part of the second one, equal to

$$\iint ik\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k$$

$$-\iint ik\Lambda(\ell-k,-\ell)\widehat{v}(\ell-k,t)\widehat{u}(-\ell,t)\widehat{v}(k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,.$$

By change of variable $(k,\ell) \mapsto (k-\ell, -\ell)$ in the first integral here above, we obtain

$$\mathrm{Re}\iint ik\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k \;=$$

$$- \iint i\ell\Lambda(k,-\ell)\widehat{v}(k,t)\widehat{u}(-\ell,t)\widehat{v}(\ell-k,t)\,\mathrm{d}\ell\,\mathrm{d}k$$

$$+ \iint ik\left(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)\right)\widehat{v}(\ell-k,t)\widehat{u}(-\ell,t)\widehat{v}(k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,,$$

where the first integral is bounded again by $\|\Lambda\|_{L^\infty}\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|v(t)\|_{L^2}^2$. As to the second integral, the factor

$$\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell) \;=\; \Lambda(k,-\ell)-\overline{\Lambda(\ell-k,-\ell)}$$

(by the reality assumption (*ii*)) is zero if the structure assumption (*iv*) is satisfied. It turns out that we can also deal with the integral under the (weaker) stability assumption (*v*). Indeed, this integral can be split into the sum of

$$\iint_{|k|\le|\ell|} ik\left(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)\right)\widehat{v}(\ell-k,t)\widehat{u}(-\ell,t)\widehat{v}(k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,,$$

whose modulus is bounded by $2\|\Lambda\|_{L^\infty}\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|v(t)\|_{L^2}^2$, and of

$$\iint_{|k|>|\ell|} ik\left(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)\right)\widehat{v}(\ell-k,t)\widehat{u}(-\ell,t)\widehat{v}(k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,.$$

This integral can be decomposed again as the sum of four integrals, each taken on a sector on which $(k,\ell)\mapsto\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)$ is smooth. These four sectors are $\{0<\ell<k\}$ and $\{-k<\ell<0\}$, and their images by the center symmetry $(k,\ell)\to(-k,-\ell)$. By the reality assumption (*ii*) it is sufficient to estimate the integrals on the first two sectors. Now for $0<\ell<k$,

$$\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell) \;=\; \Lambda(1,-\ell/k)-\Lambda(-1+\ell/k,-\ell/k)$$

by (*iii*). Since

$$\Lambda(1,0-)=\Lambda(-1,0-)$$

by (*ii*) and (*v*), we may rewrite the above equality as

$$\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell) \;=\; \Lambda(1,-\ell/k)-\Lambda(1,0-)+\Lambda(-1,0-)-\Lambda(-1+\ell/k,-\ell/k)\,,$$

and thus obtain the bound, for $0<\ell<k$,

$$|\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)| \;\le\; (\max_D|\partial_1\Lambda|+\max_I|\partial_2\Lambda|)\left|\frac{\ell}{k}\right|\,,$$

where D, I are the line segments joining respectively the points $(-1,0)$-$(0,-1)$ and $(1,0)$-$(1,-1)$. The very same bound is obtained for $-k<\ell<0$, by using (*ii*). So finally we get

$$\left|\iint_{|k|>|\ell|} ik\left(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)\right)\widehat{v}(\ell-k,t)\widehat{u}(-\ell,t)\widehat{v}(k,t)\,\mathrm{d}\ell\,\mathrm{d}k\right| \le$$

$$4\,(\max_D|\partial_1\Lambda|+\max_I|\partial_2\Lambda|)\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|v(t)\|_{L^2}^2\,.$$

This proves the L^2 estimate (2.12).

The derivation of higher order estimates is a little bit trickier but follows the same lines. (It is to be noted that no commutator estimate is required.) We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^n v(t)\|_{L^2}^2 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int k^{2n}\,|\widehat{v}(k,t)|^2\,\mathrm{d}k \;=$$

$$-2\,\mathrm{Re}\iint ik^{2n}(k-\ell)\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k\,.$$

As in the case $n=0$, we make a change of variables $(k,\ell)\mapsto(k-\ell,-\ell)$ in the integral, and leave its conjugate unchanged. This yields

$$\mathrm{Re}\iint ik^{2n}(k-\ell)\Lambda(k-\ell,\ell)\widehat{v}(k-\ell,t)\widehat{u}(\ell,t)\widehat{v}(-k,t)\,\mathrm{d}\ell\,\mathrm{d}k\;=$$

$$\iint i\left(k\,(k-\ell)^{2n}\Lambda(k,-\ell)-(k-\ell)k^{2n}\,\Lambda(\ell-k,-\ell))\right)\widehat{v}(k,t)\widehat{u}(-\ell,t)\widehat{v}(\ell-k,t)\,\mathrm{d}\ell\,\mathrm{d}k$$

To estimate this integral, the idea is to distribute in a suitable way the powers of k, ℓ, and $k-\ell$ among $\widehat{v}(-k,t)$, $\widehat{u}(\ell,t)$, and $\widehat{v}(k-\ell,t)$ respectively. Let us begin with $n=1$.

As in the case $n=0$, the contribution of the domain $|k|\leq|\ell|$ to the integral is harmless. Indeed, we have

$$\left|\iint_{|k|\leq|\ell|} ik^2(k-\ell)\Lambda(k-\ell,\ell)\widehat{v}(k-\ell)\widehat{u}(\ell)\widehat{v}(-k)\,\mathrm{d}\ell\,\mathrm{d}k\right|\;\leq$$

$$\|\Lambda\|_{L^\infty}\iint|(k-\ell)\widehat{v}(k-\ell)\,\ell\widehat{u}(\ell)\,k\widehat{v}(-k)|\,\mathrm{d}\ell\,\mathrm{d}k\;\leq\;\|\Lambda\|_{L^\infty}\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|\partial_x v\|^2_{L^2}$$

For the other part we observe that

$$\iint_{|k|>|\ell|}\left|\left(k\,(k-\ell)^2\Lambda(k,-\ell)-(k-\ell)k^2\,\Lambda(\ell-k,-\ell))\right)\widehat{v}(k,t)\widehat{u}(-\ell,t)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k\leq$$

$$\iint_{|k|>|\ell|}|\Lambda(k,-\ell)\,k\widehat{v}(k,t)\,\ell\widehat{u}(-\ell,t)\,(\ell-k)\widehat{v}(\ell-k,t)|\,\mathrm{d}\ell\,\mathrm{d}k+$$

$$\iint_{|k|>|\ell|}|k\,(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell)))\,k\widehat{v}(k,t)\widehat{u}(-\ell,t)(\ell-k)\widehat{v}(\ell-k,t)|\,\mathrm{d}\ell\,\mathrm{d}k\,,$$

where the former admits again the bound $\|\Lambda\|_{L^\infty}\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|\partial_x v\|^2_{L^2}$, and the latter can be estimated exactly as before (in the case $n=0$). This proves in turn that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x v\|^2_{L^2}\;\leq\;4\,(\|\Lambda\|_{L^\infty}\;+\;2\,\max_D|\partial_1\Lambda|+\max_I|\partial_2\Lambda|)\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|\partial_x v\|^2_{L^2}\,,$$

which together with (2.12) implies (2.13).

We observe that, as for a (local) transport equation, we have an H^1 estimate which does not require more derivatives on the coefficient (u) than the L^2 estimate. A difference with local transport equations though, is that $\|\mathcal{F}(\partial_x u)\|_{L^1}$ plays the role of the smaller norm $\|\partial_x u\|_{L^\infty}$. Nevertheless, observing that $\|\mathcal{F}(\partial_x u)\|_{L^1}$ will be bounded provided that u belongs to H^s, $s>3/2$, we can hope to deal with the well-posedness of the nonlinear Cauchy problem in these spaces, as for the classical Burgers equation. However, to avoid fractional derivatives, we shall deal with well-posedness in H^2 only.

This means we need an H^2 a priori estimate, which are going to derive now. A first, easy way consists in deducing it from the former estimates, which may written as

$$|\langle v,\mathcal{B}[u,\partial_x v]\rangle|\leq C(\Lambda)\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|v\|^2_{L^2}\,,\tag{2.15}$$

$$|\langle\partial_x v,\partial_x\mathcal{B}[u,\partial_x v]\rangle|\leq C(\Lambda)\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|\partial_x v\|^2_{L^2}\,,\tag{2.16}$$

Then by (2.7) we have

$$\left|\langle \partial_x^2 v, \partial_x^2 \mathcal{B}[u, \partial_x v]\rangle\right| \le \left|\langle \partial_x^2 v, \partial_x \mathcal{B}[\partial_x u, \partial_x v]\rangle\right| + \left|\langle \partial_x^2 v, \partial_x \mathcal{B}[u, \partial_x^2 v]\rangle\right| ,$$

where the latter term is bounded by $C(\Lambda)\, \|\mathcal{F}(\partial_x u)\|_{L^1}\, \|\partial_x^2 v\|_{L^2}^2$ thanks to (2.16), and, by Cauchy-Schwarz and (2.9), the former is bounded by $\|\Lambda\|_{L^\infty} \|\partial_x^2 v\|_{L^2}\, \|\partial_x u\|_{H^1}\, \|\partial_x v\|_{H^1}$ up to a (harmless) multiplicative constant. Therefore, up to substituting a larger positive constant (depending only on Λ) for $C(\Lambda)$,

$$\left|\langle \partial_{xx}^2 v, \partial_{xx}^2 \mathcal{B}(u, \partial_x v)\rangle\right| \le C(\Lambda)\, \|u\|_{H^2}\, \|v\|_{H^2}^2 \,, \tag{2.17}$$

which together with (2.15) and (2.16) proves (2.14).

We now prove (2.14) in a more direct (and more technical) way, which yields an additional estimate for the nonlinear equation (2.5). Proceeding as described before for higher order estimates, we have

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^2 v(t)\|_{L^2}^2 \le \\
\left|\iint i\left(k\,(k-\ell)^4 \Lambda(k,-\ell) - (k-\ell)k^4\,\Lambda(\ell-k,-\ell))\right) \widehat{v}(k,t)\widehat{u}(-\ell,t)\widehat{v}(\ell-k,t)\,\mathrm{d}\ell\,\mathrm{d}k\right| \le \\
\iint \left|((k-\ell)^3 - k^3)\Lambda(k,-\ell)\, k\widehat{v}(k,t)\, \widehat{u}(-\ell,t)\, (k-\ell)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k + \\
\iint \left|\, k^3\,(\Lambda(k,-\ell) - \Lambda(\ell-k,-\ell))\, k\widehat{v}(k,t)\, \widehat{u}(-\ell,t)\, (k-\ell)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k\,.
\end{gathered}$$

We can estimate the first integral by decomposing

$$(k-\ell)^3 - k^3 = (k-\ell)\,\ell^2 - k\,\ell^2 - 3\ell\, k\,(k-\ell)\,.$$

This yields the bound

$$\|\Lambda\|_{L^\infty}\, \left(2\,\|\mathcal{F}(\partial_x v)\|_{L^1}\, \|\partial_x^2 u\|_{L^2}\, \|\partial_x^2 v\|_{L^2} + 3\,\|\mathcal{F}(\partial_x u)\|_{L^1}\, \|\partial_x^2 v\|_{L^2}^2\right)\,.$$

Concerning the second integral, we split it again. We have

$$\begin{gathered}
\iint_{|k|\le|\ell|} \left|\, k^3\,(\Lambda(k,-\ell) - \Lambda(\ell-k,-\ell))\, k\widehat{v}(k,t)\, \widehat{u}(-\ell,t)\, (k-\ell)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \le \\
\iint \left|\, \ell^2(\ell-k+k)\,\left(\Lambda(k,-\ell) - \Lambda(\ell-k,-\ell)\right) k\widehat{v}(k,t)\, \widehat{u}(-\ell,t)\, (k-\ell)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \le \\
4\,\|\Lambda\|_{L^\infty}\, \|\mathcal{F}(\partial_x v)\|_{L^1}\, \|\partial_x^2 u\|_{L^2}\, \|\partial_x^2 v\|_{L^2}\,,
\end{gathered}$$

while

$$\begin{gathered}
\iint_{|k|>|\ell|} \left|\, k^3\,(\Lambda(k,-\ell) - \Lambda(\ell-k,-\ell))\, k\widehat{v}(k,t)\, \widehat{u}(-\ell,t)\, (k-\ell)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \le \\
4\,(\max_D |\partial_1 \Lambda| + \max_I |\partial_2 \Lambda|) \iint_{|k|>|\ell|} \left|\,(k-\ell+\ell)\,\ell\, k^2 \widehat{v}(k,t)\, \widehat{u}(-\ell,t)\, (k-\ell)\widehat{v}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \le \\
4\,(\max_D |\partial_1 \Lambda| + \max_I |\partial_2 \Lambda|)\, \left(\|\mathcal{F}(\partial_x u)\|_{L^1}\, \|\partial_x^2 v\|_{L^2}^2 + \|\mathcal{F}(\partial_x v)\|_{L^1}\, \|\partial_x^2 v\|_{L^2}\, \|\partial_x^2 u\|_{L^2}\right)\,.
\end{gathered}$$

So we recover the H^2 estimate (2.14) by using the Cauchy-Schwarz inequality to bound $\|\mathcal{F}(\partial_x w)\|_{L^1}$ by $\|w\|_{H^2}$ for $w = v$ and $w = u$. In the special case $u = v$ we obtain a more precise estimate for the nonlinear equation (2.5), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^2}^2 \le C(\Lambda)\, \|\mathcal{F}(\partial_x u)\|_{L^1}\, \|u\|_{H^2}^2\,. \tag{2.18}$$

In fact, by the very same procedure, we can obtain also the H^3 estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^3}^2 \leq C(\Lambda)\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|u\|_{H^3}^2\,. \tag{2.19}$$

Indeed, for all smooth enough solutions u of (2.5) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^3 u(t)\|_{L^2}^2 \leq$$

$$\left|\iint i\left(k\,(k-\ell)^6\Lambda(k,-\ell) - (k-\ell)k^6\,\Lambda(\ell-k,-\ell)\right)\right)\widehat{u}(k,t)\widehat{u}(-\ell,t)\widehat{u}(\ell-k,t)\,\mathrm{d}\ell\,\mathrm{d}k\right| \leq$$

$$\iint \left|((k-\ell)^5-k^5)\Lambda(k,-\ell)\,k\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k +$$

$$\iint \left|\,k^5\,(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell))\,k\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k$$

To bound the first integral we use the identity

$$(k-\ell)^5-k^5 = -(k-\ell)^2\,\ell^3 - k^2\,\ell^3 - 5\,\ell\,k^2\,(k-\ell)^2\; - 3\,k\,(k-\ell)\,\ell^3\,,$$

in which only the last term seems to be a problem (because when multiplied by $k(k-\ell)$ it yields a distribution of derivatives as $2+2+3$ instead of $1+3+3$). But of course we can bound $|k^2(k-\ell)^2\ell^3|$ by $|k||k-\ell|^3|\ell|^3 + |k-\ell||k|^3|\ell|^3$. Therefore, we find that

$$\iint \left|((k-\ell)^5-k^5)\Lambda(k,-\ell)\,k\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \lesssim$$

$$\|\Lambda\|_{L^\infty}\,\|\mathcal{F}(\partial_x u)\|_{L^1}\,\|\partial_x^3 u\|_{L^2}^2\,.$$

As regards the other integral, we observe that

$$\iint_{|k|\leq|\ell|} \left|\,k^5\,(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell))\,k\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \leq$$

$$\iint \left|\,\ell^3(\ell-k+k)^2\;(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell))\,k\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \leq$$

$$8\,\|\Lambda\|_{L^\infty}\,\|\mathcal{F}(\partial_x u)\|_{L^1}\;\|\partial_x^3 u\|_{L^2}^2\,,$$

while

$$\iint_{|k|>|\ell|} \left|\,k^5\,(\Lambda(k,-\ell)-\Lambda(\ell-k,-\ell))\,k\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \leq$$

$$4\,(\max_D|\partial_1\Lambda| + \max_I|\partial_2\Lambda|)\iint_{|k|>|\ell|}\left|\,(k-\ell+\ell)^2\,\ell\,k^3\widehat{u}(k,t)\,\widehat{u}(-\ell,t)\,(k-\ell)\widehat{u}(\ell-k,t)\right|\,\mathrm{d}\ell\,\mathrm{d}k \leq$$

$$8\,(\max_D|\partial_1\Lambda| + \max_I|\partial_2\Lambda|)\;\|\mathcal{F}(\partial_x u)\|_{L^1}\;\|\partial_x^3 u\|_{L^2}^2\,.$$

$\square$

3 Well-posedness

Once we have a priori estimates in $H^2(\mathbb{R})$, a fairly general method (see for instance [7, p. 360]) to actually prove well-posedness in $H^2(\mathbb{R})$ consists in regularizing (1.1) in such a way that the regularized problem is merely solvable by the Cauchy-Lipschitz theorem and that its solutions converge in a suitable manner to solutions of the original problem.

A simple, and natural way to regularize (1.1) is by means of Fourier multipliers. In what follows we shall use a Fourier multiplier S_ε of symbol

$$\widehat{S_\varepsilon}(\xi) = \widehat{S_1}(\varepsilon\xi)$$

with $\widehat{S_1}$ real valued, $\mathscr{C}^\infty$ with compact support, taking the value 1 at zero and of absolute value not greater than 1. Clearly, for all $\varepsilon \geq 0$, S_ε is a bounded operator on $H^s(\mathbb{R})$ for each $s \in \mathbb{R}$, with

$$\|S_\varepsilon\|_{H^s \to H^s} \leq 1\,. \tag{3.20}$$

Furthermore, for all $\varepsilon > 0$, S_ε is a regularizing operator, with

$$\|S_\varepsilon\|_{H^s \to H^{s+\sigma}} \lesssim \varepsilon^{-\sigma} \tag{3.21}$$

for all $s \in \mathbb{R}$ and $\sigma \geq 0$, and we have the error estimate

$$\|S_\varepsilon u - u\|_{H^s} \lesssim \varepsilon^\sigma\, \|u\|_{H^{s+1}} \tag{3.22}$$

for all $\varepsilon \geq 0$. Here above, the multiplicative constants hidden in the symbol $\lesssim$ depend on (s, σ) but not of ε of course.

Let us consider the following regularization of (1.1)

$$\partial_t u^\varepsilon + S_\varepsilon \mathcal{B}\,(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon) \;=\; 0\,. \tag{3.23}$$

For $\varepsilon > 0$, the mapping $u \mapsto S_\varepsilon \mathcal{B}\,(u, \partial_x S_\varepsilon u)$ is locally Lipschitz in $H^2(\mathbb{R})$, and more precisely, for all $\varepsilon \in (0, 1]$, if u and v belong to $H^2(\mathbb{R})$then

$$\begin{aligned}\|S_\varepsilon \mathcal{B}\,(u, \partial_x S_\varepsilon u) - S_\varepsilon \mathcal{B}\,(v, \partial_x S_\varepsilon v)\,\|_{H^2} &\leq \|S_\varepsilon \mathcal{B}\,(u - v, \partial_x S_\varepsilon u)\,\|_{H^2} + \|S_\varepsilon \mathcal{B}\,(v, \partial_x S_\varepsilon (u - v))\,\|_{H^2} \\ &\leq \frac{C}{\varepsilon}\,\|\Lambda\|_{L^\infty}\,(\|u\|_{H^2} \;+\; \|v\|_{H^2})\,\|u - v\|_{H^2}\,.\end{aligned}$$

Therefore, the Picard iteration scheme

$$u_0^\varepsilon := u_0\,, \quad u_{k+1}^\varepsilon : t \in [0, T^\varepsilon] \mapsto u_{k+1}^\varepsilon(t) := -\int_0^t S_\varepsilon \mathcal{B}\,(u_k^\varepsilon(\tau), \partial_x S_\varepsilon u_k^\varepsilon(\tau))\;\mathrm{d}\tau\,,\; k \in \mathbb{N}$$

is well defined and convergent in $B_R := \{u\,;\; \|u\|_{H^2} \leq R\}$ provided that $2CR\|\Lambda\|_{L^\infty} T^\varepsilon \leq \varepsilon$. This shows the existence of a solution $u^\varepsilon \in \mathscr{C}^1(0, T^\varepsilon; H^2(\mathbb{R}))$ of (3.23) such that $u^\varepsilon(0) = u_0$. This solution is unique and depends continuously on u_0 by Gronwall's lemma. Indeed, if $v^\varepsilon \in \mathscr{C}^1(0, T^\varepsilon; B_R)$ is another solution of (3.23) we have

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{H^2} \;\leq\; \|u^\varepsilon(0) - v^\varepsilon(0)\|_{H^2} \;+\; \frac{2CR}{\varepsilon}\,\|\Lambda\|_{L^\infty} \int_0^t \|u^\varepsilon(\tau) - v^\varepsilon(\tau)\|_{H^2}\,\mathrm{d}\tau\,,$$

hence

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{H^2} \;\leq\; (1 + \mathrm{e}^{\frac{2CRt}{\varepsilon}\,\|\Lambda\|_{L^\infty}})\;\|u^\varepsilon(0) - v^\varepsilon(0)\|_{H^2} \;\leq\; (1 + \mathrm{e})\,\|u^\varepsilon(0) - v^\varepsilon(0)\|_{H^2}$$

for $t \le T^\varepsilon$ (by assumption on T^ε). This shows continuous dependence on initial data for (3.23), and uniqueness within the ball B_R. Unconditional uniqueness follows from a classical connectedness argument.

Let us now redefine T^ε as the *maximal* time of existence of the solution of (3.23) with initial data $u^\varepsilon(0) = u_0 \in B_R$. By the construction hereabove we have $T^\varepsilon \ge \varepsilon/(2CR\|\Lambda\|_{L^\infty})$. It remains to show that T^ε is positively bounded by below when $\varepsilon \to 0$, and that u^ε converges to a solution of (1.1) in H^2.

As a first step, we show that $\|u^\varepsilon(t)\|_{H^2}$ is bounded independently of $\varepsilon > 0$ and $t \in [0,T]$ for some positive T. This relies on the a priori estimates (2.15) (2.16) (2.17). Indeed, since S_ε is a self-adjoint operator and commutes with ∂_x,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon\|_{H^2}^2 \;=\; -2\,\langle S_\varepsilon u^\varepsilon, \mathcal{B}(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon)\rangle - 2\,\langle \partial_x S_\varepsilon u^\varepsilon, \partial_x \mathcal{B}(u^\varepsilon, S_\varepsilon \partial_x u^\varepsilon)\rangle$$

$$-2\,\langle \partial_{xx}^2 S_\varepsilon u^\varepsilon, \partial_{xx}^2 \mathcal{B}(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon)\rangle,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon\|_{H^2}^2 \;\le 6C(\lambda)\,\|u^\varepsilon\|_{H^2}^3\,,$$

and after integration

$$\|u^\varepsilon(t)\|_{H^2} \;\le\; \frac{\|u_0\|_{H^2}}{1 - 3C(\lambda)t\|u_0\|_{H^2}}$$

for all $t \in [0, T^\varepsilon)$ such that $t < 1/(3C(\lambda)R)$. As a consequence, T^ε cannot be lower than $1/(3C(\lambda)R)$ (otherwise, denoting $R^\varepsilon := R/(1 - 3C(\Lambda)T^\varepsilon R)$, we could extend u^ε behond T^ε, restarting from $u^\varepsilon(t_0^\varepsilon)$ with $t_0^\varepsilon = T^\varepsilon - \varepsilon/(4CR^\varepsilon\|\Lambda\|_{L^\infty})$ as initial data, which would contradict the fact that T^ε is maximal). From now on, we choose $T < 1/(3C(\lambda)R)$. By the argument above, $T^\varepsilon > T$ for all $\varepsilon > 0$, and $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $\mathscr{C}(0,T;H^2(\mathbb{R}))$.

The heart of the matter then consists in showing that $(u^\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ satisfies the Cauchy criterion in $\mathscr{C}(0,T;L^2(\mathbb{R}))$. For $0 < \nu \le \varepsilon$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u^\nu\|_{L^2}^2 \;=\; & -2\,\langle u^\varepsilon - u^\nu, S_\varepsilon\mathcal{B}(u^\varepsilon - u^\nu, \partial_x S_\varepsilon u^\varepsilon)\rangle \\ & -2\,\langle u^\varepsilon - u^\nu, S_\varepsilon\mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\rangle \\ & -2\,\langle u^\varepsilon - u^\nu, (S_\varepsilon - S_\nu)\mathcal{B}(u^\nu, \partial_x S_\nu u^\nu)\rangle\,.\end{aligned}$$

We are going estimate these three terms separately. By Cauchy-Schwarz and (2.8) (2.9) we can estimate the first and last terms

$$|2\,\langle u^\varepsilon - u^\nu, S_\varepsilon\mathcal{B}(u^\varepsilon - u^\nu, \partial_x S_\varepsilon u^\varepsilon)\rangle| \lesssim \|\Lambda\|_{L^\infty}\,\|u^\varepsilon\|_{H^2}\,\|u^\varepsilon - u^\nu\|_{L^2}^2\,,$$

$$|2\,\langle u^\varepsilon - u^\nu, (S_\varepsilon - S_\nu)\mathcal{B}(u^\nu, \partial_x S_\nu u^\nu)\rangle| \lesssim \|\Lambda\|_{L^\infty}\,\|u^\varepsilon - u^\nu\|_{L^2}\,\varepsilon\,\|u^\nu\|_{H^1}\,\|u^\nu\|_{H^2}\,.$$

As to the middle term, using again that S_ε is self-adjoint, we can split it as

$$-2\,\langle S_\varepsilon u^\varepsilon - S_\nu u^\nu, \mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\rangle + 2\,\langle (S_\varepsilon - S_\nu)u^\nu, \mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\rangle\,.$$

By Cauchy-Schwarz and (2.8) again we have

$$|2\,\langle (S_\varepsilon - S_\nu)u^\nu, \mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\rangle| \lesssim \|\Lambda\|_{L^\infty}\,\varepsilon\,\|u^\nu\|_{H^1}^2\,(\|u^\varepsilon\|_{H^1} + \|u^\nu\|_{H^1})\,,$$

and by (2.15),

$$|2\,\langle S_\varepsilon u^\varepsilon - S_\nu u^\nu, \mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\rangle| \lesssim C(\Lambda)\,\|\partial_x u^\nu\|_{H^1}\,\|S_\varepsilon u^\varepsilon - S_\nu u^\nu\|_{L^2}^2$$

$$\lesssim C(\Lambda)\,\|\partial_x u^\nu\|_{H^1}\,\left(\|u^\varepsilon - u^\nu\|_{L^2} + \varepsilon\,\|u^\nu\|_{H^1}\right)^2\,.$$

Adding these estimates altogether, and using that $\|u^\varepsilon\|_{H^2}$, $\|u^\nu\|_{H^2}$, are uniformly bounded by R we obtain the (rather crude) estimate (for some modified constant $\widetilde{C}(\Lambda)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u^\nu\|_{L^2}^2 \le \widetilde{C}(\Lambda)\, R\, \|u^\varepsilon - u^\nu\|_{L^2}^2 + \widetilde{C}(\Lambda)\, R^3\, \varepsilon\,,$$

which yields by integration, using that $u^\varepsilon(0) = u^\nu(0)$,

$$\|u^\varepsilon(t) - u^\nu(t)\|_{L^2}^2 \le \widetilde{C}(\Lambda)\, R^3\, \varepsilon\, \mathrm{e}^{\widetilde{C}(\Lambda)\, R t}\,.$$

Therefore, u^ε is convergent in $\mathscr{C}(0,T;L^2(\mathbb{R}))$ as ε goes to zero. It remains to show some additional regularity for its limit u. In fact, since $(u^\varepsilon(t))$ is uniformly bounded in $H^2(\mathbb{R})$, we have $u(t) \in H^2(\mathbb{R})$ for all $t \in [0,T]$. By $L^2 - H^2$ interpolation, this implies that u^ε converges to u in $\mathscr{C}(0,T;H^s(\mathbb{R}))$ for all $s \in [0,2)$. Now, by interpolation between (2.8) and (2.9), we have

$$\|\mathcal{B}(v,w)\|_{H^s} \lesssim \|\Lambda\|_{L^\infty}\, \|v\|_{H^1}\, \|w\|_{H^s}\,, \tag{3.24}$$

for all $s \in [0,1]$ and $v \in H^1$, $w \in H^s$. This will enable us to show that $S_\varepsilon \mathcal{B}\left(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon\right)$ converges to $\mathcal{B}(u, \partial_x u)$ in $\mathscr{C}(0,T;H^s(\mathbb{R}))$ for $s \in [0,1)$. Indeed, we have the pointwise time estimate

$$\begin{gathered}\|S_\varepsilon \mathcal{B}\left(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon\right) - \mathcal{B}(u,\partial_x u)\|_{H^s} \le \\ \varepsilon^{(1-s)/2}\, \|\mathcal{B}\left(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon\right)\|_{H^1} + \|\mathcal{B}\left(u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon\right) - \mathcal{B}(u,\partial_x u)\|_{H^s} \lesssim \\ \varepsilon^{(1-s)/2}\, \|u^\varepsilon\|_{H^1}\, \|u^\varepsilon\|_{H^2} + \|\mathcal{B}\left(u^\varepsilon, \partial_x(S_\varepsilon u^\varepsilon - u)\right)\|_{H^s} + \|\mathcal{B}\left(u^\varepsilon - u, \partial_x u\right)\|_{H^s} \lesssim \\ \varepsilon^{(1-s)/2}\, \|u^\varepsilon\|_{H^1}\, \|u^\varepsilon\|_{H^2} + \|u^\varepsilon - u\|_{H^1}\, \|u\|_{H^{s+1}}\,.\end{gathered}$$

(For simplicity we have included $\|\Lambda\|_{L^\infty}$ in the $\lesssim$ symbol.) Therefore, being the limit of $\partial_t u^\varepsilon$ in the sense of distributions, $\partial_t u = -\mathcal{B}(u, \partial_x u)$ belongs to $\mathscr{C}(0,T;H^s(\mathbb{R}))$. This shows that u is in $\mathscr{C}^1(0,T;H^s(\mathbb{R}))$ for all $s \in [0,1)$. Then we can prove that u is the unique solution of (1.1) in $\mathscr{C}(0,T;H^{s+1}(\mathbb{R})) \cap \mathscr{C}^1(0,T;H^s(\mathbb{R})) \cap L^\infty(0,T;H^2(\mathbb{R}))$ with initial data u_0. For, if v were another one, we would have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u - v\|_{L^2}^2 = -2\,\langle u - v, \mathcal{B}(u - v, u)\rangle - 2\,\langle u - v, \mathcal{B}(v, \partial_x(u - v))\rangle \lesssim K\, \|u - v\|_{L^2}^2\,.$$

Here above we have used Cauchy-Schwarz to estimate the first term, (2.15) for the second, and K is a uniform bound for $(\|u\|_{H^2} + \|v\|_{H^2})$ on the time interval $[0,T]$. So we have by integration

$$\|u(t) - v(t)\|_{L^2}^2 \le \mathrm{e}^{Kt}\|u(0) - v(0)\|_{L^2}^2\,.$$

We already know that u belongs to $L^\infty(0,T;H^2(\mathbb{R}))$. To conclude that u is actually in $\mathscr{C}(0,T;H^2(\mathbb{R}))$ we invoke weak topology arguments. Since u^ε converges to u in $\mathscr{C}(0,T;H^s(\mathbb{R}))$ for all $s \in [0,2)$, by density of the dual $H^{-s}(\mathbb{R})$ of $H^s(\mathbb{R})$ in $H^{-2}(\mathbb{R})$, we see that $u^\varepsilon(t)$ converges uniformly on $[0,T]$ to u in $H^2_w(\mathbb{R})$, the Sobolev space $H^2(\mathbb{R})$ equipped with the weak topology. By a similar argument, for all $t_0 \in [0,T]$, $u(t)$ tends to $u(t_0)$ in $H^2_w(\mathbb{R})$ when t goes to t_0, which implies in particular

$$\liminf_{t\to t_0} \|u(t)\|_{H^2} \ge \|u(t_0)\|_{H^2}\,.$$

Therefore, to prove the strong limit

$$\lim_{t\to t_0} \|u(t) - u(t_0)\|_{H^2} = 0\,,$$

it suffices to prove

$$\limsup_{t\to t_0} \|u(t)\|_{H^2} \le \|u(t_0)\|_{H^2}\,.$$

Now, as shown before (substituting $t-t_0$ for t),

$$\|u^\varepsilon(t)\|_{H^2} \le \frac{\|u(t_0)\|_{H^2}}{1-3C(\lambda)(t-t_0)\|u(t_0)\|_{H^2}}$$

for all $\varepsilon > 0$ and $t \in [t_0, T]$, hence

$$\|u(t)\|_{H^2} \le \liminf_{\varepsilon\searrow 0} \|u^\varepsilon(t)\|_{H^2} \le \frac{\|u(t_0)\|_{H^2}}{1-3C(\lambda)(t-t_0)\|u(t_0)\|_{H^2}},$$

and finally

$$\limsup_{t\searrow t_0} \|u(t)\|_{H^2} \le \|u(t_0)\|_{H^2}.$$

The inequality for $t \nearrow t_0$ can be obtained in a similar way by reversing time.

We have thus proved the following.

Theorem 3.1 *Assuming the kernel Λ is $\mathscr{C}^1$ outside the lines $k=0$, $\ell=0$, and $k+\ell=0$, has $\mathscr{C}^1$ continuations to the sectors delimited by these lines, and satisfies (ii), (iii), (v), for all $u_0 \in H^2(\mathbb{R})$ there exists $T>0$ and a unique solution $u \in \mathscr{C}(0,T;H^2(\mathbb{R})) \cap \mathscr{C}^1(0,T;H^1(\mathbb{R}))$ of (1.1).*

Continuous dependence with respect to initial data in H^2 demands a little more work.

Theorem 3.2 *Under the assumptions of Theorem 3.1 the mapping*

$$\begin{array}{lcl} H^2(\mathbb{R}) & \to & \mathscr{C}(0,T;H^2(\mathbb{R})) \\ u_0 & \mapsto & u\,, \quad \textit{solution of } (1.1) \textit{ such that } u(0)=u_0\,, \end{array}$$

is continuous.

To prove this result, we shall use a trick originally introduced by Bona and Smith for KdV [4] (also see [2]), and regularize initial data by means of S_{ε^β} for a suitable $\beta > 0$. In this respect we shall make the further assumption that $\widehat{S_\varepsilon}$ equals one on the interval $[-1/\varepsilon, 1/\varepsilon]$. This implies that for all $s \in \mathbb{R}$, $\sigma \ge 0$, for all $u \in H^s(\mathbb{R})$,

$$\|S_\varepsilon u - u\|_{H^{s-\sigma}} = o(\varepsilon^\sigma)\,, \tag{3.25}$$

and moreover, for any sequence $(u^n)_{n\in\mathbb{N}}$ converging to u in $H^s(\mathbb{R})$,

$$\varepsilon^{-\sigma}\|S_\varepsilon u - u\|_{H^{s-\sigma}} = o(1)\,, \quad \text{uniformly with respect to } n\,. \tag{3.26}$$

Lemma 3.1 *We assume that the Fourier multiplier S_ε satisfies* (3.20), (3.21), (3.22), (3.25), *and* (3.26). *We take a $\beta \in (0,1/2)$. Then, under the assumptions of Theorem 3.1, for all $R>0$ there exists $T>0$ such that for all $u_0 \in H^2(\mathbb{R})$ of norm not greater than R, for all $\varepsilon > 0$, the Cauchy problem for* (3.23) *and initial data $u^\varepsilon(0) = S_{\varepsilon^\beta} u_0$ admits a unique solution $u^\varepsilon \in \mathscr{C}(0,T;H^3(\mathbb{R}))$. Furthermore, we have*

$$\|u^\varepsilon\|_{\mathscr{C}(0,T;H^2(\mathbb{R}))} = \mathcal{O}(1)\,, \quad \|u^\varepsilon\|_{\mathscr{C}(0,T;H^3(\mathbb{R}))} = \mathcal{O}(\varepsilon^{-\beta})\,,$$

and u^ε converges in $\mathscr{C}(0,T;H^2(\mathbb{R}))$ to a solution u of (1.1) *such that $u(0)=u_0$, with the following rate in H^1:*

$$\|u^\varepsilon - u\|_{H^1} = \mathcal{O}(\varepsilon^{1-\beta})\,.$$

.

Proof. By a slight modification of the argument used above for the Cauchy problem with non-regularized initial data, we easily see that for all $\varepsilon > 0$, the Cauchy problem with regularized initial $u^\varepsilon(0) = S_{\varepsilon^\beta} u_0$ admits a unique local solution in H^3, the maximal time of existence T^ε being at least of order of $\varepsilon^{1+\beta}/\|u_0\|_{H^2}$ (like $\varepsilon/\|S_{\varepsilon^\beta} u_0\|_{H^3}$). In addition, by integration of

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon\|_{H^2}^2 \le 6C(\Lambda)\,\|u^\varepsilon\|_{H^2}^3\,,$$

we obtain

$$\|u^\varepsilon(t)\|_{H^2} \le \frac{\|u^\varepsilon(0)\|_{H^2}}{1-3C(\Lambda)t\|u^\varepsilon(0)\|_{H^2}} \le \frac{\|u_0\|_{H^2}}{1-3C(\Lambda)t\|u_0\|_{H^2}}$$

for all $t \in [0, T^\varepsilon)$, hence $T^\varepsilon \ge 1/(3C(\Lambda)R$ if $\|u_0\|_{H^2} \le R$. From now on we take $T < 1/(3C(\Lambda)R)$, in such a way that $\|u^\varepsilon(t)\|_{H^2}$ is uniformly bounded for $t \in [0,T]$ and $\varepsilon > 0$, which also implies a uniform bound for $\|\mathcal{F}(\partial_x u^\varepsilon)\|_{L^\infty(0,T;L^1(\mathbb{R}))}$.

Now, revisiting the proof of (2.12) (2.13) (2.18) (2.19) (just using that $|\widehat{S_\varepsilon}| \le 1$) we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon\|_{H^3}^2 \le C(\Lambda)\,\|\mathcal{F}(\partial_x u^\varepsilon)\|_{L^1}\,\|u^\varepsilon\|_{H^3}^2\,.$$

By integration, this yields

$$\|u^\varepsilon(t)\|_{H^3}^2 \le \|u^\varepsilon(0)\|_{H^3}^2\,\mathrm{e}^{C(\Lambda)\|\mathcal{F}(\partial_x u^\varepsilon)\|_{L^1((0,T)\times\mathbb{R})}}\,,$$

hence

$$\|u^\varepsilon\|_{\mathscr{C}(0,T;H^3(\mathbb{R}))} \lesssim \|S_{\varepsilon^\beta} u_0\|_{H^3} \lesssim \varepsilon^{-\beta}\,\|u_0\|_{H^2}\,. \tag{3.27}$$

Thanks to this estimate and the uniform bound of $\|u^\varepsilon\|_{\mathscr{C}(0,T;H^2(\mathbb{R}))}$, say R', we can now show that $(u^\varepsilon)_{\varepsilon>0}$ satifies the Cauchy criterion not only in $\mathscr{C}(0,T;L^2(\mathbb{R}))$ (as done before) but also in $\mathscr{C}(0,T;H^2(\mathbb{R}))$. For $m \in \mathbb{N}$, $0 < \nu \le \varepsilon$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^m(u^\varepsilon - u^\nu)\|_{L^2}^2 = \;& -\langle \partial_x^m(u^\varepsilon - u^\nu), S_\varepsilon \partial_x^m \mathcal{B}(u^\varepsilon - u^\nu, \partial_x S_\varepsilon u^\varepsilon)\rangle \\ & -\langle \partial_x^m(S_\varepsilon u^\varepsilon - S_\nu u^\nu), \partial_x^m(\mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu)))\rangle \\ & +\langle (S_\varepsilon - S_\nu)\partial_x^m u^\nu, \partial_x^m(\mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu)))\rangle \\ & -\langle \partial_x^m(u^\varepsilon - u^\nu), (S_\varepsilon - S_\nu)\partial_x^m \mathcal{B}(u^\nu, \partial_x S_\nu u^\nu)\rangle\,.\end{aligned}$$

For convenience we call I_i^m, $i = 1,2,3,4$, the terms above. We first concentrate on the case $m = 1$. By Cauchy-Schwarz and (2.9),

$$|I_1^1| \le \|\partial_x(u^\varepsilon - u^\nu)\|_{L^2}\,\|\mathcal{B}(u^\varepsilon - u^\nu, \partial_x S_\varepsilon u^\varepsilon)\|_{H^1} \lesssim \|\partial_x u^\varepsilon\|_{H^1}\,\|u^\varepsilon - u^\nu\|_{H^1}^2\,.$$

By the energy estimate (2.16) and the error estimate (3.22),

$$|I_2^1| \lesssim \|\mathcal{F}(\partial_x u^\nu)\|_{L^1}\,\|\partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu)\|_{L^2}^2 \lesssim \|\mathcal{F}(\partial_x u^\nu)\|_{L^1}\,(\|\partial_x(u^\varepsilon - u^\nu)\|_{L^2}^2 + \varepsilon^2\|\partial_x u^\nu\|_{H^1}^2)\,.$$

By Cauchy-Schwarz, (2.9), (3.22), and (3.27),

$$|I_3^1| \lesssim \|(S_\varepsilon - S_\nu)\partial_x u^\nu\|_{L^2}\,\|\mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\|_{H^1}$$

$$\lesssim \varepsilon^2\|\partial_x u^\nu\|_{H^1}\,\|u^\nu\|_{H^1}\,(\|u^\nu\|_{H^3} + \|u^\varepsilon\|_{H^3}) \lesssim \varepsilon^{2-\beta}\,\|\partial_x u^\nu\|_{H^1}\,\|u^\nu\|_{H^1}\,\|u_0\|_{H^2}\,.$$

By Cauchy-Schwarz, (2.10), and (3.27),

$$|I_4^1| \lesssim \varepsilon\,\|\partial_x(u^\varepsilon - u^\nu)\|_{L^2}\,\|\mathcal{B}(u^\nu, \partial_x S_\nu u^\nu)\|_{H^2} \lesssim \varepsilon^{1-\beta}\,\|\partial_x(u^\varepsilon - u^\nu)\|_{L^2}\,\|u^\nu\|_{H^2}\,\|u_0\|_{H^2}\,.$$

By adding all four estimates we get (recalling also the L^2 estimate)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u^\nu\|_{H^1}^2 \lesssim R' \, \|u^\varepsilon - u^\nu\|_{H^1}^2 \, + \, (R')^3 \, \varepsilon^2 + (\varepsilon^{2(1-\beta)} + \varepsilon^{2-\beta}) \, R' R^2 \, .$$

Therefore, by integration we obtain

$$\|u^\varepsilon(t) - u^\nu(t)\|_{H^1}^2 \, \leq \, \mathrm{e}^{C't} \, (\|u^\varepsilon(0) - u^\nu(0)\|_{H^1}^2 \, + \, (R')^2 \, \varepsilon^2 + (\varepsilon^{2(1-\beta)} + \varepsilon^{2-\beta}) \, R^2 \,) \, ,$$

where C' is proportional to R'. Since $\|u^\varepsilon(0) - u^\nu(0)\|_{H^1} \lesssim \varepsilon \, \|u_0\|_{H^2}$ by (3.22), we receive a uniform estimate

$$\|u^\varepsilon - u^\nu\|_{\mathscr{C}(0,T;H^1(\mathbb{R}))} \, = \, \mathcal{O}(\varepsilon^{1-\beta}) \, . \tag{3.28}$$

Let us now turn to the estimate of $\|u^\varepsilon(t) - u^\nu(t)\|_{H^2}$. By Cauchy-Schwarz, (2.10), and (3.28)

$$\begin{aligned} |I_1^2| &\leq \|\partial_x^2(u^\varepsilon - u^\nu)\|_{L^2} \, \|\mathcal{B}(u^\varepsilon - u^\nu, \partial_x S_\varepsilon u^\varepsilon)\|_{H^2} \\ &\lesssim \|u^\varepsilon - u^\nu\|_{H^2} \, (\|\partial_x u^\varepsilon\|_{H^2} \, \|u^\varepsilon - u^\nu\|_{H^1} \, + \, \|\partial_x u^\varepsilon\|_{H^1} \, \|u^\varepsilon - u^\nu\|_{H^2} \,) \\ &\lesssim \|\partial_x u^\varepsilon\|_{H^1} \, \|u^\varepsilon - u^\nu\|_{H^2}^2 \, + \, \mathcal{O}(\varepsilon^{1-2\beta}) \, \|u^\varepsilon - u^\nu\|_{H^2} \, . \end{aligned}$$

By the energy estimate (2.17), the error estimate (3.22), and the uniform bound (3.27)

$$\begin{aligned} |I_2^2| \lesssim \|\mathcal{F}(\partial_x u^\nu)\|_{L^1} \, \|\partial_x^2(S_\varepsilon u^\varepsilon - S_\nu u^\nu)\|_{L^2}^2 &\lesssim \|\mathcal{F}(\partial_x u^\nu)\|_{L^1} \, (\|\partial_x(u^\varepsilon - u^\nu)\|_{L^2}^2 + \varepsilon^2 \|\partial_x u^\nu\|_{H^2}^2) \\ &\lesssim \|\partial_x u^\nu\|_{H^1} \, (\|\partial_x(u^\varepsilon - u^\nu)\|_{L^2}^2 + \varepsilon^{2(1-\beta)} \|u_0\|_{H^2}^2) \, . \end{aligned}$$

By Cauchy-Schwarz, (2.9), (3.22), and (3.27),

$$\begin{aligned} |I_3^2| &\lesssim \|(S_\varepsilon - S_\nu)\partial_x^2 u^\nu\|_{L^2} \, \|\mathcal{B}(u^\nu, \partial_x(S_\varepsilon u^\varepsilon - S_\nu u^\nu))\|_{H^2} \\ &\lesssim \varepsilon \|\partial_x^2 u^\nu\|_{H^1} \, (\, \|u^\nu\|_{H^1} \, \|(S_\varepsilon u^\varepsilon - S_\nu u^\nu)\|_{H^3} \, + \, \|u^\nu\|_{H^2} \, \|(S_\varepsilon u^\varepsilon - S_\nu u^\nu)\|_{H^2} \,) \\ &\lesssim \varepsilon \, \|u^\nu\|_{H^2}^2 \, (\, \|u^\varepsilon - u^\nu\|_{H^2} + \varepsilon^{-\beta} \|u_0\|_{L^2} \,) \, . \end{aligned}$$

The most 'dangerous' term is in I_4^2. It can dealt with by first integrating by part, which leads to

$$I_4^2 = - \, \langle \partial_x^3(u^\varepsilon - u^\nu), (S_\varepsilon - S_\nu)\partial_x \mathcal{B}(u^\nu, \partial_x S_\nu u^\nu) \rangle \, ,$$

hence by Cauchy-Schwarz, (2.10), and (3.27),

$$|I_4^2| \lesssim \varepsilon \, \|\partial_x^3(u^\varepsilon - u^\nu)\|_{L^2} \, \|\mathcal{B}(u^\nu, \partial_x S_\nu u^\nu)\|_{H^2} \lesssim \varepsilon^{1-2\beta} \, \|u_0\|_{H^2}^2 \, \|u^\nu\|_{H^2} \, .$$

By summation of these four estimates with the estimates obtained for first order derivatives, we finally arrive at an inequality of the form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u^\nu\|_{H^2}^2 \lesssim \|u^\varepsilon - u^\nu\|_{H^2}^2 \, + \, \mathcal{O}(\varepsilon^{1-2\beta}) \, .$$

As a consequence, we get

$$\|u^\varepsilon - u^\nu\|_{\mathscr{C}(0,T;H^2)} \lesssim \|u^\varepsilon(0) - u^\nu(0)\|_{H^2} \, + \, \mathcal{O}(\varepsilon^{1/2-\beta}) \, = \, o(1) \, .$$

by (3.25) applied to u_0, $s = 2$, and $\sigma = 0$. $\square$

Proof of Theorem 3.2. It amounts to proving that for any sequence $(u_0^n)_{n\in\mathbb{N}}$ tending to u_0 in $H^2(\mathbb{R})$, the solutions u_n of the Cauchy problems

$$\partial_t u_n + \mathcal{B}[u_n, \partial_x u_n] = 0\,, \quad u_n(0) = u_0^n \tag{3.29}$$

go to the solution of

$$\partial_t u + \mathcal{B}[u, \partial_x u] = 0\,, \quad u(0) = u_0\,. \tag{3.30}$$

Let us take $(u_0^n)_{n\in\mathbb{N}}$ tending to u_0 in $H^2(\mathbb{R})$. We first observe that since $(u_0^n)_{n\in\mathbb{N}}$ is bounded in H^2, by Lemma 3.1 the solution u_n of (3.29) in H^2 is well-defined on some interval $[0,T]$ independent of n, as well as the solutions u_n^ε and u^ε of the regularized Cauchy problems

$$\partial_t u_n^\varepsilon + S_\varepsilon \mathcal{B}[u_n^\varepsilon, \partial_x S_\varepsilon u_n^\varepsilon] = 0\,, \quad u_n(0) = S_\varepsilon u_0^n\,, \tag{3.31}$$

$$\partial_t u^\varepsilon + S_\varepsilon \mathcal{B}[u^\varepsilon, \partial_x S_\varepsilon u^\varepsilon] = 0\,, \quad u^\varepsilon(0) = S_\varepsilon u_0\,. \tag{3.32}$$

Furthermore, by Lemma 3.1 we also have that $\|u^\varepsilon - u\|_{\mathscr{C}(0,T;H^2)}$ goes to zero, and revisiting its proof with the help of (3.26), we also find that $\|u_n^\varepsilon - u_n\|_{\mathscr{C}(0,T;H^2)}$ goes to zero uniformly in n. We can now conclude by an $\varepsilon/3$-(or more appropriately here an $\eta/3$)-argument. Indeed, for all $t\in[0,T]$, for all $n\in\mathbb{N}$, for all $\varepsilon>0$,

$$\|u_n(t) - u(t)\|_{H^2} \;\leq\; \|u_n(t) - u_n^\varepsilon(t)\|_{H^2} \;+\; \|u_n^\varepsilon(t) - u^\varepsilon(t)\|_{H^2} \;+\; \|u^\varepsilon(t) - u(t)\|_{H^2}\,.$$

For $\eta>0$, there exists ε_0 such that for all $\varepsilon\in(0,\varepsilon_0)$, for all $t\in[0,T]$, for all $n\in\mathbb{N}$,

$$\|u_n(t) - u_n^\varepsilon(t)\|_{H^2} + \|u^\varepsilon(t) - u(t)\|_{H^2} \;\leq 2\eta/3\,.$$

If we choose an $\varepsilon\in(0,\varepsilon_0)$, as we have seen in the proof of Theorem 3.1,

$$\|u_n^\varepsilon(t) - u^\varepsilon(t)\|_{H^2} \;\leq\; C_\varepsilon\, \|u_n^\varepsilon(0) - u^\varepsilon(0)\|_{H^2} \;\leq\; C_\varepsilon\, \|u_n(0) - u(0)\|_{H^2}$$

by (3.20). So $\|u_n(t) - u(t)\|_{H^2}$ can be made less than η for n large enough. □

Another result that comes out from the proof of Lemma 3.1 is the following blow-up criterion, which generalizes the well-known blow-up criterion for the classical inviscid Burgers equation ($\lim_{t\nearrow T}\|\partial_x u\|_{L^1(0,T;L^\infty(\mathbb{R}))} = +\infty$ if T is a finite, maximal time of existence).

Corollary 3.1 *Under the assumptions of Theorem 3.1, if $u\in\mathscr{C}(0,T;H^2(\mathbb{R}))$ is a solution of* (1.1) *such that $\mathcal{F}(\partial_x u)$ belongs to $L^1((0,T)\times\mathbb{R})$ then u can be extended beyond T.*

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